An IRT-based Parameterization for Conditional Probability Tables

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Educational Testing Models

- Usually some collection of latent variables representing target of assessment: *proficiency model,* θ_i
  - Usually positively correlated
- Each item (task) presented to examinee has one or more observable outcome variables X_{ij}
- *Evidence model* for Task j is $Pr(X_{ij} \mid \theta_i)$

Monotonic and Parametric CPTs

- Both parent (proficiency) and child (observable) variables are ordered categorical
- Want CPTs to be monotonically increasing: i.e., higher values of proficiency imply higher probability of better outcomes
- Because proficiency variables are correlated, may only be a few observations for some rows of CPTs

Discrete IRT (2PL) model

- Imagine a case with a single parent and a binary (correct/incorrect) child.
- Map states of parent variable onto a continuous scale: *effective theta,* $\tilde{\theta}_m$
- Plug into IRT equation to get conditional probability of "correct"

$$\Pr(Y_j = 1|X = m) = \mathrm{logit}^{-1}\left[1.7a_j(\tilde{\theta}_m - b_j)\right]$$

  - a_j – discrimination parameter
  - b_j – difficulty parameter
  - *1.7* – Scaling constant (makes logistic curve look like normal ogive)

Multivariate Models: Combination Rules

- For Multiple Parents, assign each parent j an effective theta at each level k, $\tilde{\theta}_{j,k}$
- Combine Using a Combination Rule (Structure Function)

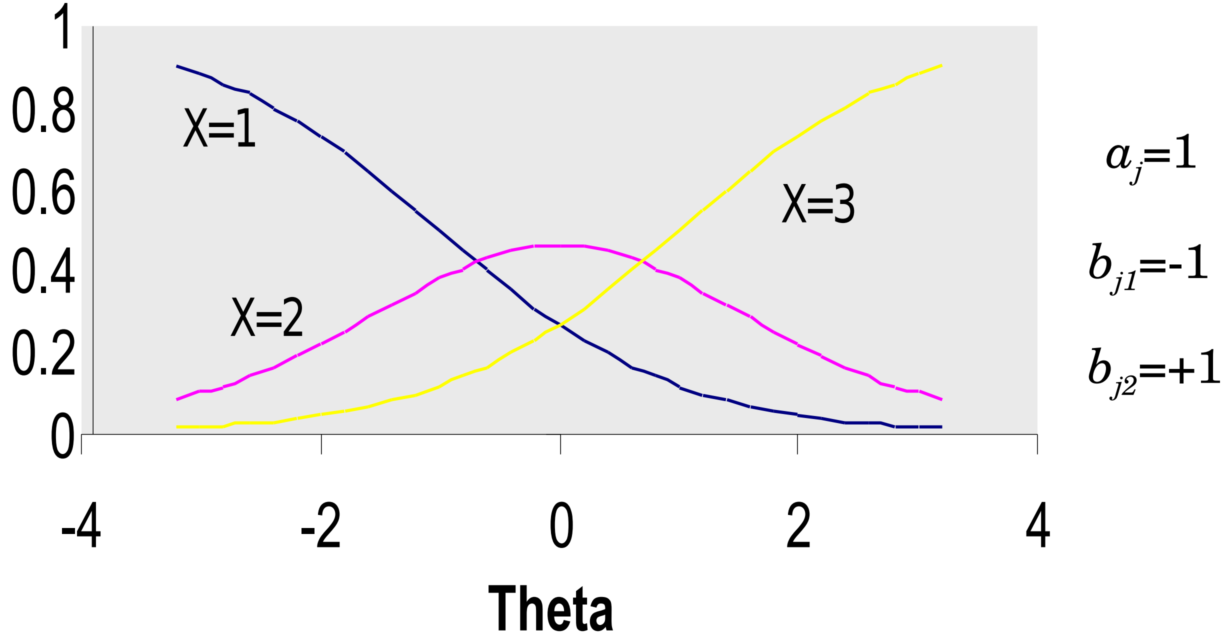
$$s(\tilde{\theta}_{1,k_1}, \ldots, \tilde{\theta}_{J,k_J})$$

- Possible Structure Functions:
  - Compensatory = average
  - Conjunctive = min
  - Disjunctive = max
  - Inhibitor; e.g. level k^* on θ_1: $\begin{cases} s(\tilde{\theta}_{1,k_1}, \ldots, \tilde{\theta}_{J,k_J}) & \text{if } k_1 > k^* \\ \tilde{\theta}_0 & \text{if } k_1 \le k^* \end{cases}$
    where $\tilde{\theta}_0$ is some low value.

DiBello--Samejima Models

- Single parent version
- Map each level of parent state to "effective theta" on IRT (N(0,1)) scale, $\tilde{\theta}_k$
- Now plug into Samejima graded response model to get probability of outcome
- Uses standard IRT parameters, "difficulty" and "discrimination"
- DiBello--Normal model uses regression model rather than graded response

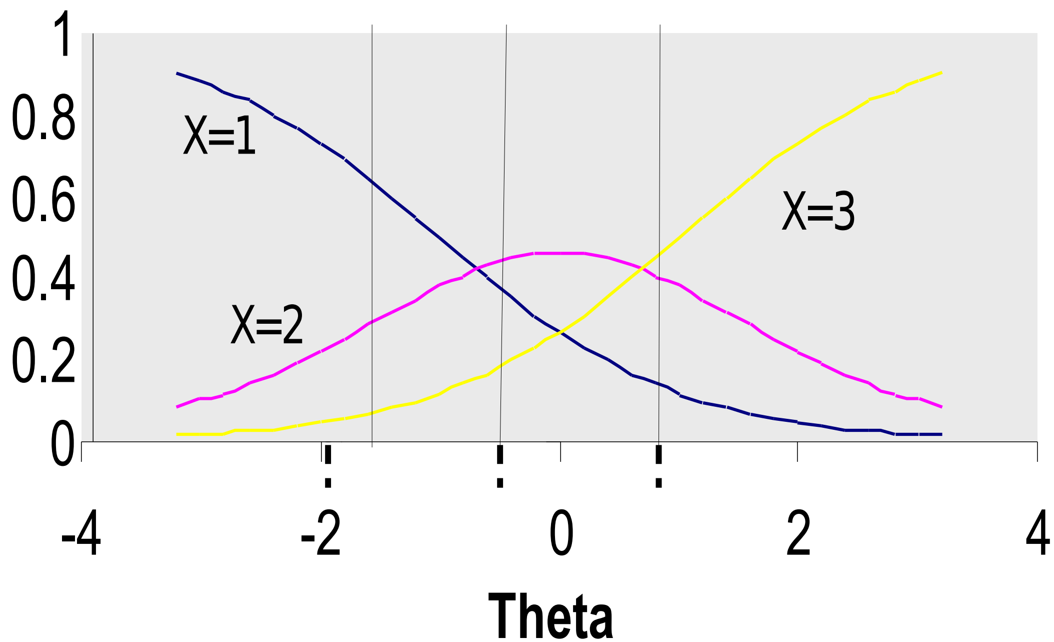
Samejima's Graded Response Model



Samejima's (1969) psychometric model for graded responses:

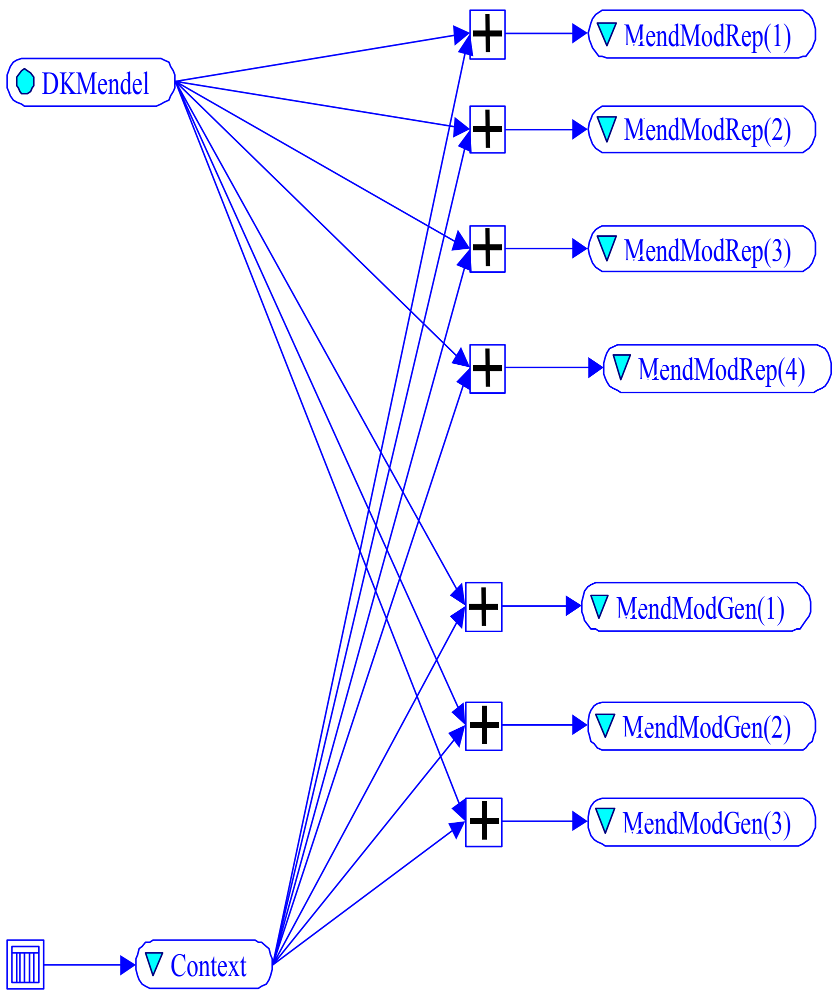
$$
\begin{aligned}
\Pr(X_{i,j} \geq k|\theta_i) &= logit^{-1}(a_j\theta_i + b_{j,k}) \\
\Pr(X_{i,j} = k|\theta_i) &= \Pr(X_{i,j} \geq k|\theta_i) - \Pr(X_{i,j} \geq k-1|\theta_i)
\end{aligned}
$$

The “Effective θ” Method (2): Conditional Probabilities for Three θ’s



θ	X=1 (Poor)	X=2 (Okay)	X=3 (Good)
Low= -1.8	.70	.25	.05
Med= -.4	.35	.40	.25
High= 1.0	.10	.40	.50

Example (Biomass)



DKMendel is the student-model variable that determines probabilities of response to the several observable variables in the Mode of Inheritance chart.

Context is a parent that induces conditional dependence among these observations, for reasons other than the *DKMendel* (e.g., did not understand what was required in task).

Effective Thetas for Compensatory Relationship

$\tilde{\theta}_{j,k}$ equally spaced normal quantiles

$$a_{S1} = 1 \qquad a_{\text{Context}} = .75 \qquad b_j = -1$$

S1	Context	S1.theta	Context.theta	Effective.theta
High	Familiar	0.97	0.67	2.04
Medium	Familiar	0.00	0.67	1.36
Low	Familiar	−0.97	0.67	0.67
High	Unfamiliar	0.97	−0.67	1.33
Medium	Unfamiliar	0.00	−0.67	0.64
Low	Unfamiliar	−0.97	−0.67	−0.04

Effective Theta to CPT

Introduce new parameter d_{inc} as spread between difficulties in Samejima model

$b_{i,Full} = b_j + d_{inc}/2$ $b_{j,Partial} = b_j - d_{inc}/2$

Conditional probability table for d_{inc} = 1 is then…

S1	Context	Effective.theta	Full	Partial	None
High	Familiar	2.04	0.73	0.15	0.12
Medium	Familiar	1.36	0.62	0.20	0.18
Low	Familiar	0.67	0.39	0.24	0.37
High	Unfamiliar	1.33	0.50	0.23	0.27
Medium	Unfamiliar	0.64	0.50	0.23	0.27
Low	Unfamiliar	−0.04	0.39	0.24	0.37

Limitation of Graded Response

- Samejima's graded response model requires curves to be parallel.
- Slope parameters must be the same (intercepts increasing)
- Combination functions must be the same

Generalized Partial Credit Model

- Muraki (1992)
- Focuses on state transitions

$$\Pr(X \geq m+1 | X \geq m)$$

- *Can use different slopes, sets of parents and combination rules for different state transitions*
- Graded Response, and Partial Credit are examples of *link functions* that go from linear predictor to probabilities

CPTtools framework

Building a CPT requires three steps:

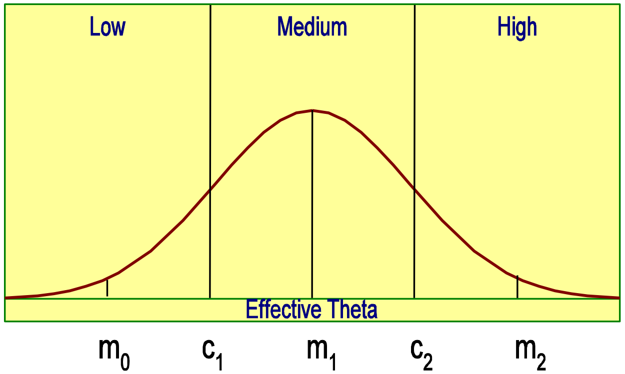
1. Map each parent state into a *effective theta* for that parent
2. Combine the parent effective thetas to an effective theta for each row of the CPT using one (or more) *combination rules*
   - Combination rules generally take one or more (often one for each parent variable) *discrimination parameters* which weight the parent variable contributions (log alphas)
   - Combination rules generally take one or more *difficulty parameters* (often one for each state of the child variable) which shift the average probability of a correct response (betas)
3. Map the effect theta for each row into a conditional probability of seeing each state using a *link function*
   - Link functions can take a scaling parameter. (link scale)

Parent level effective thetas

- Effective theta scale is a logit scale corresponds to mean 0 SD 1 in a “standard” population.
- Want the effective theta values to be equally spaced on this scale
- Want the marginal distribution implied by the effective thetas to be uniform (unit of the combination operator)
- What the effective theta transformation to be effectively invertible (this is reason to add the 1.7 to the IRT equation).

Equally spaced quantiles of the normal distribution

- Suppose variable has M states: $0,\ldots,M-1$
- Want the midpoint of the interval going from probability m/M to $(m+1)/M$.
- Solution is to map state m onto $\Phi^{-1}\left(\frac{m+1/2}{M}\right)$



- R code: `qnorm((1:M)-.5)/M)`

Combination Rules

- Compensatory – more of one skill compensates for lack of another
- Conjunctive – weakest skill dominates relationship
- Disjunctive – strongest skill dominates relationship
- Inhibitor – minimum threshold of Skill 1 needed, then Skill 2 takes over (special case of conjuctive)
- Offset Conjunctive – like conjunctive model, but with separate *b's* for each parent instead of separate *a's*
- Offset Disjunctive – like disjunctive rule, but with separate *b's* for each parent instead of separate *a's*.

Compensatory Rule

- Weighted average of inputs
- Weights are (as we often want the weights to be positive we often use as the parameter).

$$\tilde{\theta} = \frac{1}{\sqrt{K}} \sum \alpha_{k,s} \theta_{km_k} - \beta_{js}$$

- s is state of child variable
- Factor $1/\sqrt{K}$ is a variance stabilization term (makes variance stay the same as number of parents changes)

Conjunctive and Disjunctive rules

- Same setup, except replace sum with max and variance stabilization term is no longer needed:
- Conjunctive: $\tilde{\theta} = \min \alpha_{ks}\theta_{km_k} - \beta_{js}$
- Disjunctive: $\tilde{\theta} = \max \alpha_{ks}\theta_{km_k} - \beta_{js}$
- Inhibitor:

$$\tilde{\theta} = \begin{cases} \alpha_{2s}\theta_{2m_2} - \beta_{js}, & \mathrm{m}_1 > m_1^* \\ \alpha_{2s}\theta_{2,0} - \beta_{js}, & otherwise \end{cases}$$

Offset Conjunctive and Disjunctive

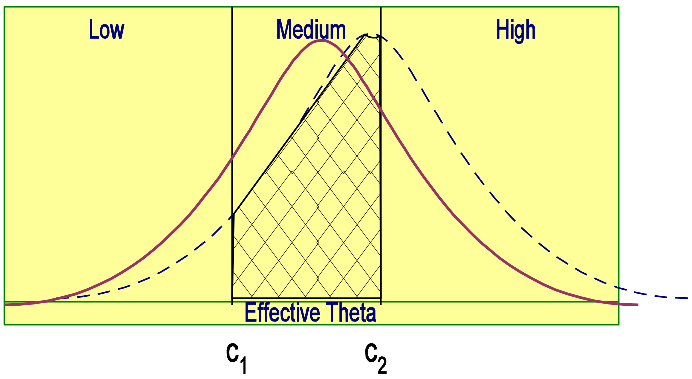
- Separate slopes doesn't really make sense for conjunctive and disjunctive models
- Separate intercepts, i.e., a different difficulty for each parent variable, does.
- Multiple betas, one alpha
- Conjunctive: $\tilde{\theta} = \alpha_{js} \min(\theta_{km_k} - \beta_{ks})$
- Disjunctive: $\tilde{\theta} = \alpha_{js} \max(\theta_{km_k} - \beta_{ks})$

Link functions

- Graded Response model
  - Models for each value of *s*
  - In order to keep the curves from crossing, discrimination parameters must be the same for all *s*
- Normal (Regression) model
  - Effective theta is mean predictor
  - Add a residual variance (link scale parameter)
  - Calculate probabilities that value falls into certain regions
- Generalized partial credit model
  - Models state transitions
  - Does not need the discrimination parameters to be the same
  - Does not even need the combination rules to be the sam

Normal Link function

- As with effective theta transformation, start by dividing theta region up into intervals
  - Equally spaced
  - Spaced to achieve a certain marginal distribution for Y
- Calculate offset curve:
  - mean is effective theta
  - SD, σ, is *link scale parameter*
- Conditional probabilities:
  - area under curve between cut points



Conjunctive-Normal model

- This is essentially a regression

$$R^2 = \frac{\sum_{k=1}^{K} \alpha_k^2/K}{\sigma^2 + \sum_{k=1}^{K} \alpha_k^2/K}$$

- Note: If child value is a proficiency variable, this is a *latent variable regression*. Correlation should be higher than you think.

Generalized Partial Credit Link

- Set up a series of conditional probabilities:

$$P_{js|s-1}(\tilde{\theta}_i) = \Pr(Y_{ij} \geq s | Y_{ij} \geq s-1, \tilde{\theta}_i) = \text{logit}^{-1}(1.7 Z_{js}(\tilde{\theta}_i))$$

- Probability of *Y* being in State *s* is:

$$\Pr(V_{ij} = s | \tilde{\theta}_i) = \frac{\prod_{r=0}^{s} P_{jr|r-1}(\tilde{\theta}_i)}{C},$$

where *C* is a normalization constant.

- Can convert the products to sums:

$$\Pr(V_{ij} = s | \tilde{\theta}_i) = \frac{\exp(1.7 \sum_{r=0}^{s} Z_{jr}(\tilde{\theta}_i))}{\sum_{R=0}^{S_j} \exp(1.7 \sum_{r=0}^{R} Z_{jr}(\tilde{\theta}_i))} .$$

Discrete Partial Credit Model (DPC)

- $Z()$ is the combination rule (structure function)
- $Z_{jr}()$ describes how skills combine to make transition between state $r\text{-}1$ and r.
- $Z_{j0}() = 0$
- Although functional form is commonly taken as the same for all states, it does not need to be!
- This allows us to model different cognitive processes at different steps

Example: Math Word Problem

- Based on unpublished analysis by Cocke and Guo (personal communication 2011-07-11)
- Next Generation Sunshine State Standards Benchmark, MA.6.A.5.1, “Use equivalent forms of fractions, decimals, and percents to solve problems" (NGSSS, 2013)
- Sample problem:

  John scored 75% on a test and Mary has 8 out of 12 correct on the same test. Each test item is worth the same amount of points. Who has the better score?

Scoring Rubric

Score Point	Description	Skills Required
0	Null response or off track	None
1	Recognizes 75% and 8/12 as key elements	Mathematical Language
2	Converts two fractions to a common form	Convert Fractions
3	Makes the correct comparison	Compare Fraction & Mathematical Language

Model Refinement

- Collapse categories 2 and 3 as very few 2's observed in practice
- Combine *Convert fractions* and *Compare fractions* into *Fraction manipulation*
- Need two combination rules:
  - $0 \rightarrow 1$: Only one skill relevant. Can use any rule, choose compensatory because it is easiest to work with. Do selection by setting discrimination for *fraction manipulation* to 0.
  - $1 \rightarrow 2$: Both skills necessary, but inhibitor model: only a minimal level of mathematical language is necessary.

Effective Thetas and Z's

Mathematical Language	$\theta_{i'1}$	*Manipulate Fractions*	$\theta_{i'2}$	$Z_{j0}(\theta_{i'})$	$Z_{j1}(\theta_{i'})$	$Z_{j2}(\theta_{i'})$
`high`	+0.97	`high`	+0.97	0	+1.47	+0.70
`high`	+0.97	`medium`	0.00	0	+1.47	-0.25
`high`	+0.97	`low`	-0.97	0	+1.47	-1.22
`medium`	0.00	`high`	+0.97	0	+0.50	+0.77
`medium`	0.00	`medium`	0.00	0	+0.50	-0.25
`medium`	0.00	`low`	-0.97	0	+0.50	-1.22
`low`	-0.97	`high`	+0.97	0	-0.47	-1.22
`low`	-0.97	`medium`	0.00	0	-0.47	-1.22
`low`	-0.97	`low`	-0.97	0	-0.47	-1.22

Conditional Probability table

Mathematical Language	*Manipulate Fractions*	$\Pr(X_{ij} = 0 \mid pa(X_{ij}))$	$\Pr(X_{ij} = 1 \mid pa(X_{ij}))$	$\Pr(X_{ij} = 2 \mid pa(X_{ij}))$
high	high	0.019	0.229	0.752
high	medium	0.047	0.576	0.377
high	low	0.068	0.828	0.104
medium	high	0.083	0.195	0.722
medium	medium	0.205	0.480	0.314
medium	low	0.275	0.644	0.081
low	high	0.664	0.299	0.038
low	medium	0.664	0.299	0.038
low	low	0.664	0.299	0.038

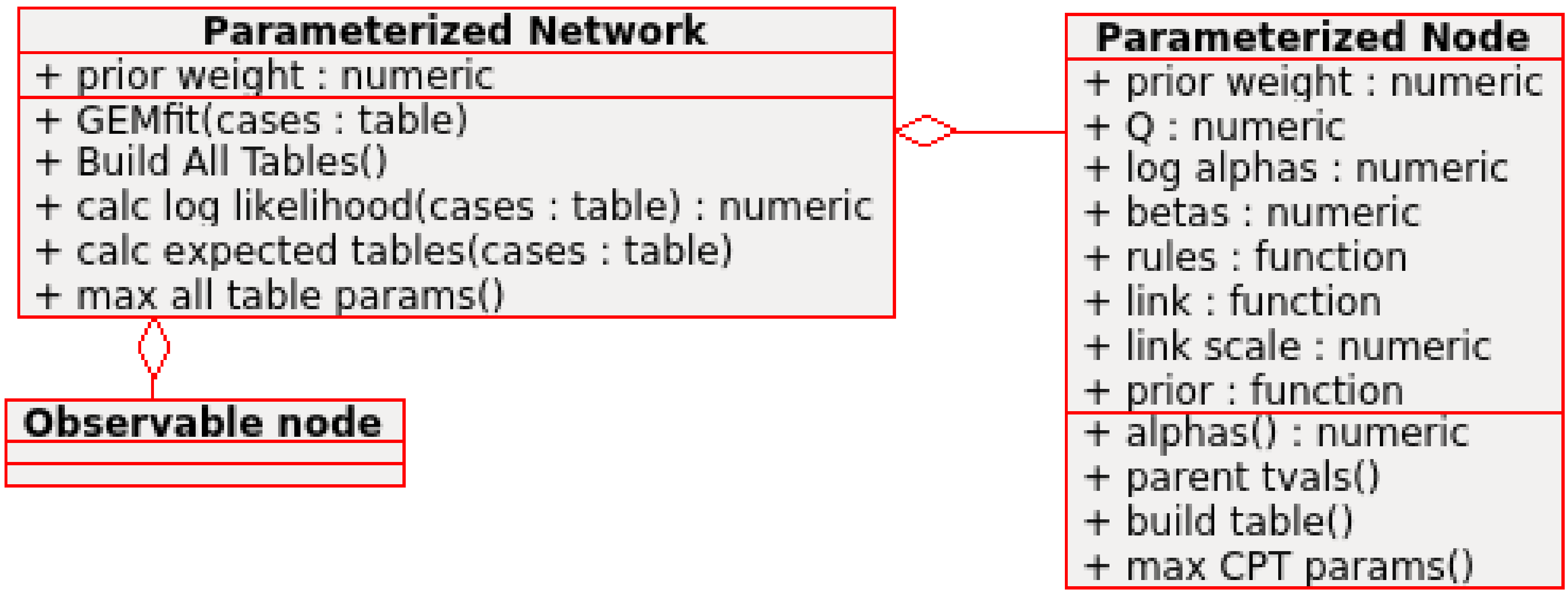
Local Q-matrix

- In GPC models transitions can use a subset of variables.
- Q is a logical matrix with rows corresponding to state transitions, and columns to parent variables
  - True if parent is relevant for that transition
- Takes advantage of R logical subscripts
- Q=TRUE is shorthand for all variables relevant for all transitions

Open Implementation Protocols in R

- R is a functional language, so functions (or list of functions) can be passed as arguments and stored as fields in objects.
  - CPTtools implementation allows link and combination rules to be passed in as functions
- R has an object oriented layer, so generic functions can be specialized for implementations
- Use rather loose S3 class system, which allows building new object oriented classes on top of existing RNetica implementation

Object Model



Lists and Vectors of Parameters

- R supports vectors (same type) and lists (any type)
- *Vectors* are used to indicate replication based on number of parameters (slope or intercept)
- *Lists* are used to indicate replication based on state transition (intercepts and slope and combination rules under GPC link)

Generalized EM algorithm

- E-step – Calculate expected value of sufficient statistics
  - Sufficient statistics in this case are the tables of counts of parent variable and child
  - Many BN packages (e.g., Netica) provide built-in EM algorithm supporting hyper-Dirichlet (unparameterized) model
  - Expected value of sufficient statistic is CPT output from this algorithm, weighted by row counts (Netica calls this Node experience)
  - Don't need to run internal EM algorithm to conclusion, one step should be fine.
- M-step find parameter values that maximize sufficient statistic
  - Can do this node by node
  - Don't need to run to convergence (generalized EM algorithm).

GEMfit

1) calcExpTables – calls internal EM algorithm (with case data) to perform E-step
2) maxAllTableParams – finds new parameters for each Pnode
3) BuildAllTables – Rebuilds the tables, and sets the weight to priorWeight
4) calcPnetLLike – Calculated the log-likelihood of data

Algorithm ends when change in log-likelihood is less than tolerance

All these functions are generic, so can be customized for different BN packages

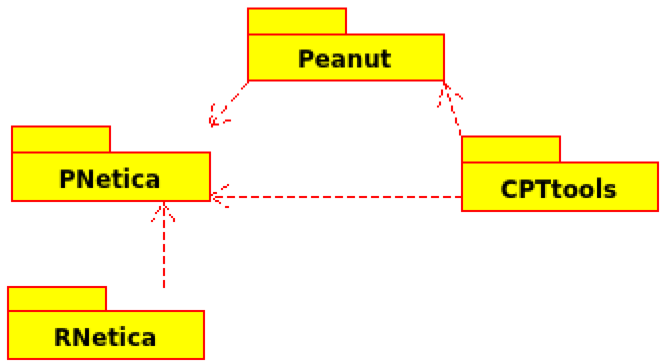
Tuning parameters

- priorWeight given to Pnet CPTs in E-step
- Number of iterations taken in E-step (1 should be sufficient)
- Number of iterations taken in M-step (5 seems good)
- Convergence Tolerance

Parameter Recovery

- Depends on number of cases
- Depends on how well latent variables are identified in those cases (amount of evidence for the latent variables): *test length*

Package Structure: Minimizing dependence on Netica



- CPTtools – Basic calculation routines, BN implementation independent
- Peanut – OO layer for Pnet/Pnode classes
- RNetica – A specific BN implementation (Netica bound in R)
- PNetica – Peanut implementation in RNetica

Availability

- http://pluto.coe.fsu.edu/RNetica
- RNetica requires Netica API license (for non-trivial examples)
- Other Bayes net package would need to support:
  - an EM learning function for hyper-Dirchlet models
  - specifying hyper-Dirichlet priors for each CPT
  - recovering the hyper-Dirichlet posteriors after running the internal EM algorithm.