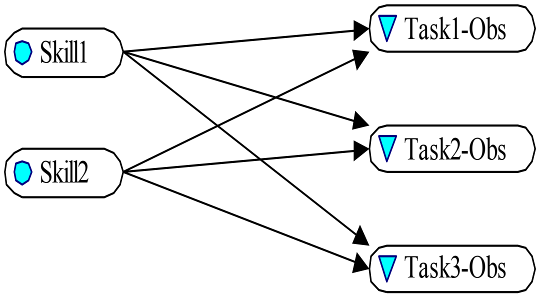
# Learning CPTs

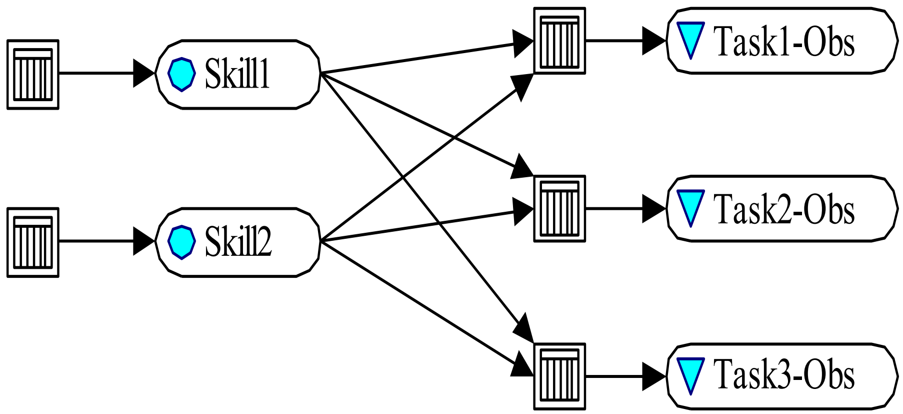
Thanks to Bob Mislevy for letting me use some of the slides from his class.

# First Layer




- A simple model with two skills and 3 observables

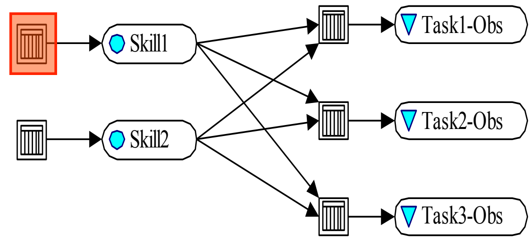
# Distributions and Variables



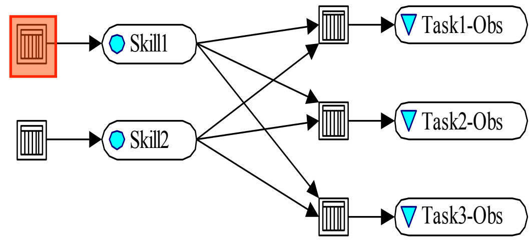

- Variables (values are person specific)
- *Distributions* provide probabilities for variables

# Different People, Same Distributions

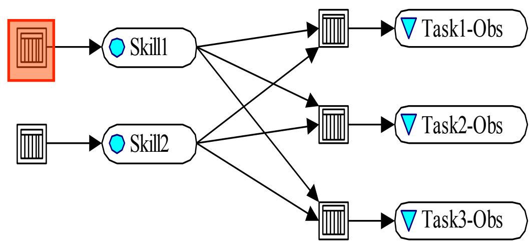
Student 1




Student 2

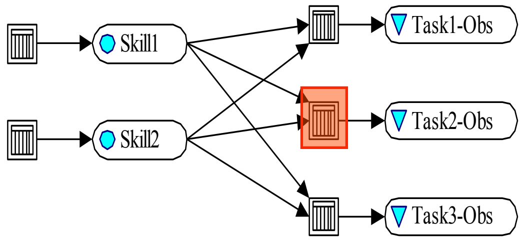



Student 3

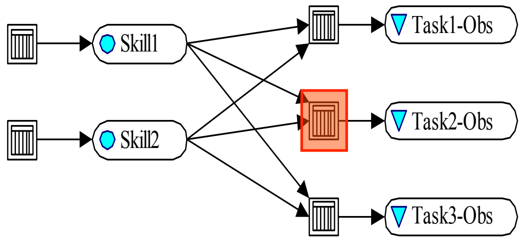


# Different People, Same Distributions

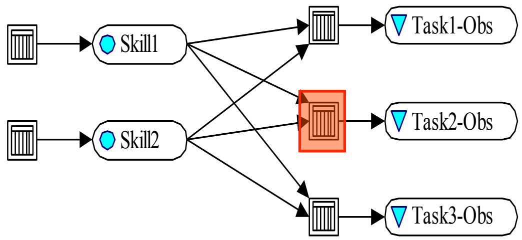
Student 1




Student 2

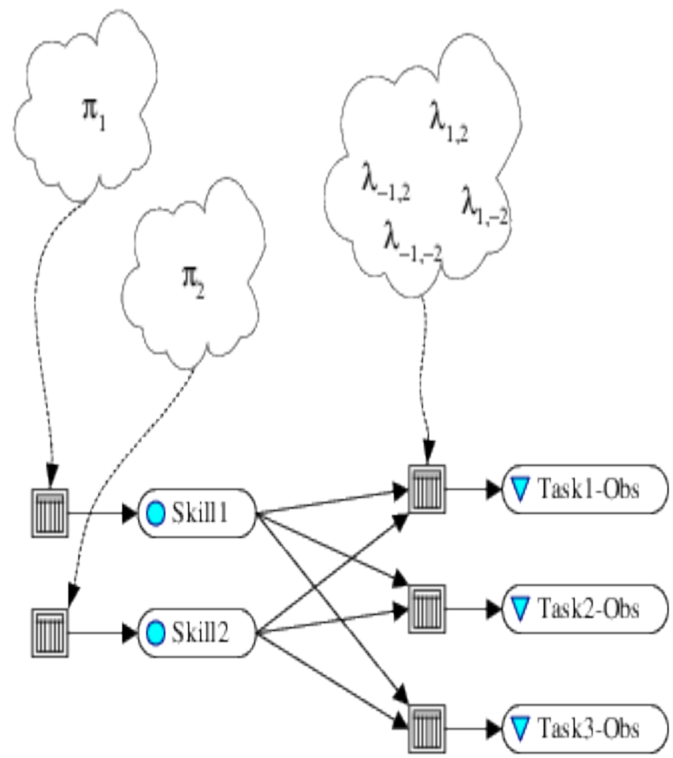



Student 3



# Second Layer

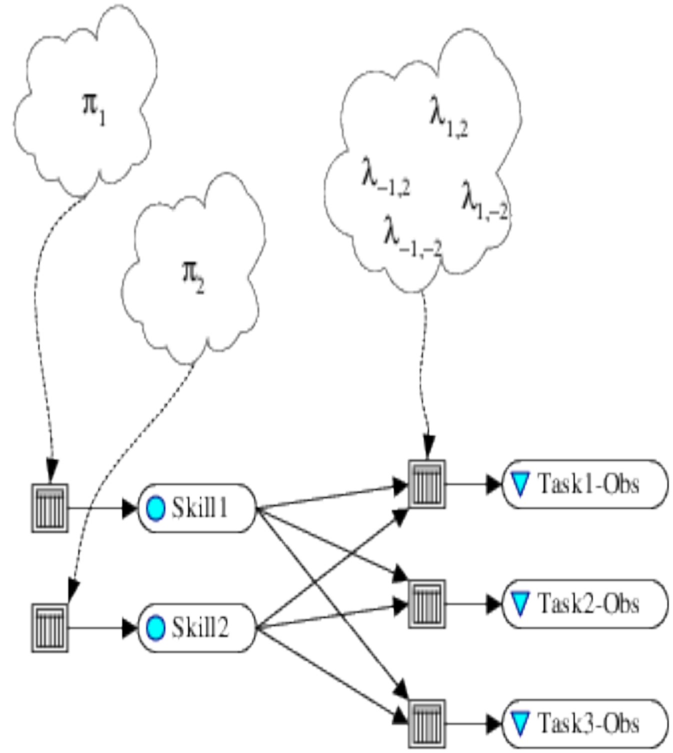
- Distributions have Parameters
- Parameters are the same across all people
- Parameters drop down into first layer to do person specific computations (e.g., scoring)




Probability distributions of parameters are called *Laws*

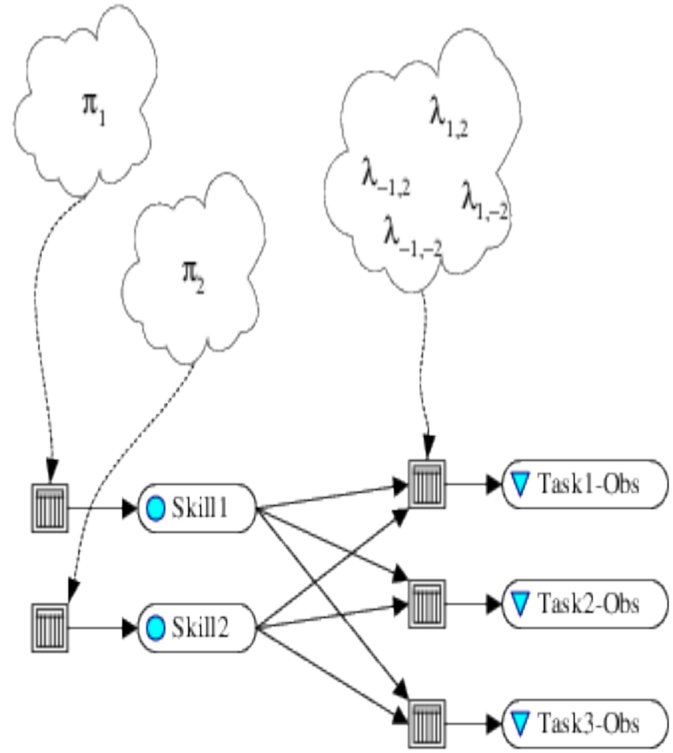
# Second Layer

- Distributions have Parameters
- Parameters are the same across all people
- Parameters drop down into first layer to do person specific computations (e.g., scoring)




Probability distributions of parameters are called *Laws*

# Second Layer




$\pi_1 = \Pr(\bigcirc\text{Skill1})$

$\pi_1 = \Pr(\bigcirc\text{Skill2})$

$\lambda_{1,2} = \Pr(\triangledown\text{Task1} - \text{obs}|\bigcirc\text{Skill1}, \bigcirc\text{Skill2})$

$\lambda_{-1,2} = \Pr(\triangledown\text{Task1} - \text{obs}|\neg\bigcirc\text{Skill1}, \bigcirc\text{Skill2})$

$\lambda_{1,-2} = \Pr(\triangledown\text{Task1} - \text{obs}|\bigcirc\text{Skill1}, \neg\bigcirc\text{Skill2})$

$\lambda_{-1,-2} = \Pr(\triangledown\text{Task1} - \text{obs}|\neg\bigcirc\text{Skill1}, \neg\bigcirc\text{Skill2})$

# Speigelhalter And Lauritzen (1990)

- *Global Parameter Independence* – parameters of laws for different CPTs are independent
- *Local Parameter Independence* – parameters for laws for different rows of CPTs are independent

Under these two assumptions, the natural conjugate law of a Bayesian network is a *hyper-Dirichlet law*, a law where the probabilities on each row of each CPT follow a Dirichlet law.

Abusing the definition, we say that a CPT for which each rows is given an independent Dirichlet law follows a *hyper-Dirichlet distribution* (really should be law).

# A closer look at the binomial distribution

- **Binomial.** For counts of successes in binary trials, each with probability p, in n independent trials. E.g., n coin flips, with p the common probability of heads.

The variable

Count of successes

$$p(r|\pi,n) \propto \pi^{r}(1-\pi)^{n-r}$$

Count of failures

The "success probability" parameter

The failure probability

The success probability

We will be using this as a likelihood in an example of the use of conjugate distribution

# A closer look at the Beta distribution

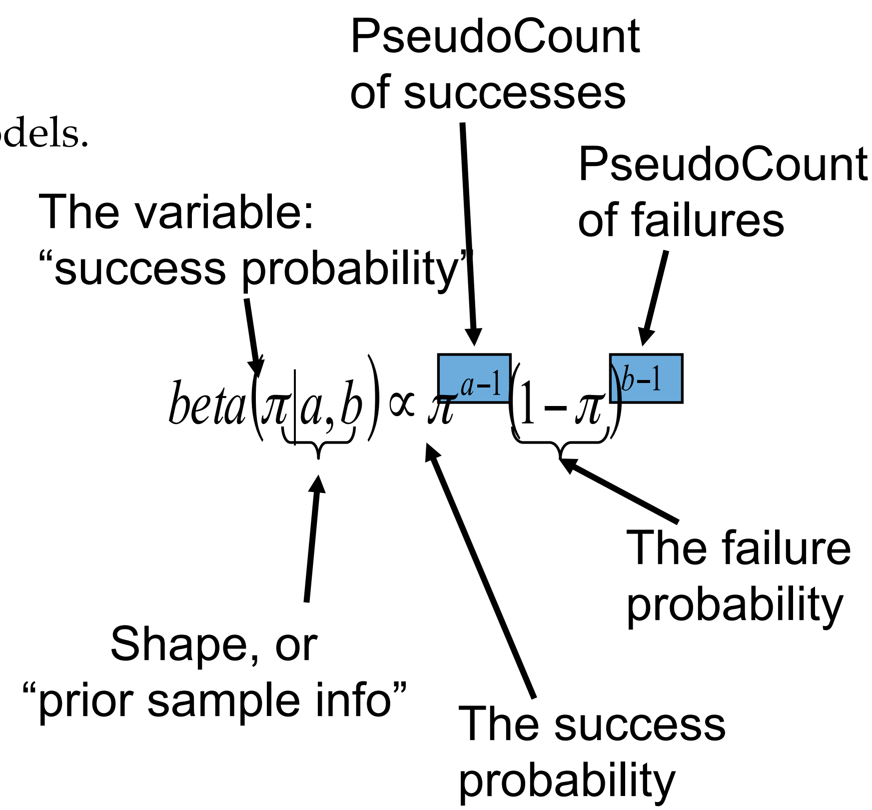
- **Beta.** Defined on [0,1]. Conjugate prior for the probability parameter in Bernoulli & binomial models.

```
p ~ dbeta(a,b)
```

Mean(p): $\frac{a}{a+b}$

Variance(p): $\frac{ab}{(a+b)^2(a+b+1)}$

Mode(p): $\frac{a-1}{a+b-2}$

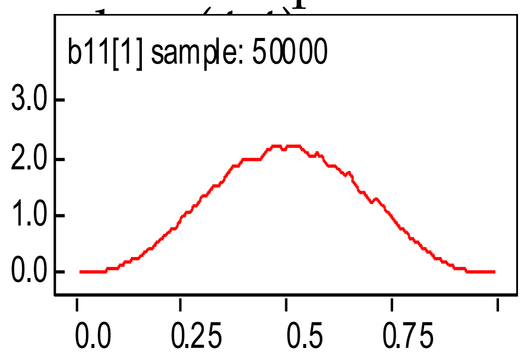


# An example with a continuous variable:

## A beta-binomial example--the Prior Distribution

- The prior distribution:

  Let's suppose we think it is more likely that the coin is close to fair, so π is probably nearer to .5 than it is to either 0 or 1. We don't have any reason to think it is biased toward either heads or tails, so we'll want a prior distribution that is symmetric around .5. We're not real sure about what π might be--say about as sure as only 6 observations. This corresponds to 3 pseudo-counts of H and 3 of T, which, if we want to use a beta distribution to express this belief, corres



# An example with a continuous variable:
## A beta-binomial example--the Prior Distribution

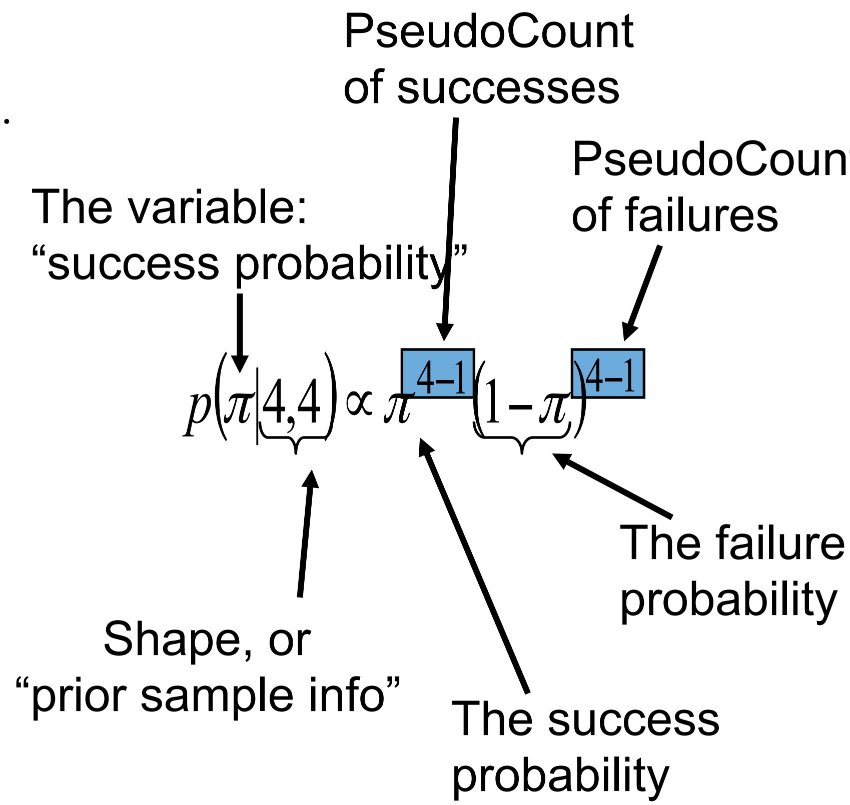
- **Beta.** Defined on [0,1]. Conjugate prior for the probability parameter in Bernoulli & binomial models.

```
π ~ dbeta(4,4)
```

Mean(π): $\frac{4}{4+4} = .5$

Variance(π): $\frac{4 \cdot 4}{(4+4)^2(4+4+1)} = .028$

Mode(π): $\frac{4-1}{4+4-2} = .5$



# An example with a continuous variable:
## A beta-binomial example--the Likelihood

- The likelihood:

  Next we will flip the coin ten times. Assuming the same true (but unknown to us) value of π is in effect for each of ten independent trials, we can use the binomial distribution to model the probability of getting any number of heads: i.e.,

$$p(r|\pi,10) \propto \pi^{r}(1-\pi)^{10-r}$$

The variable

Count of *observed* successes

Count of *observed* failures

The "success probability" parameter

The failure probability

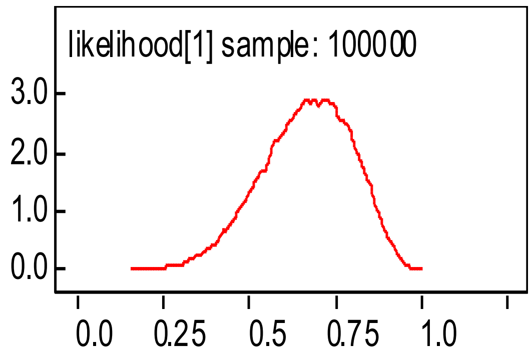
The success probability

# An example with a continuous variable:
## A beta-binomial example--the Likelihood

- The likelihood:

  We flip the coin ten times, and observe 7 heads; i.e., r=7. The likelihood is obtained now using the same form as in the preceding slide, except now r is fixed at 7 and we are interested in the relative value of this function at different possible values of $\pi$:

$$p(7|\pi,10) \propto \pi^7 (1-\pi)^3$$



# An example with a continuous variable:
## Obtaining the posterior by Bayes Theorem

General form:

posterior likelihood prior

$$p(y \mid x^*) \propto p(x^* \mid y) \quad p(y)$$

In our example, 7 plays the role of x*, and p plays the role of y. Before normalizing:

$$p(\pi \mid r = 7) \propto \left[\pi^7 (1-\pi)^3\right]\left[\pi^{4-1}(1-\pi)^{4-1}\right]$$

$$= \left[\pi^{10}(1-\pi)^6\right]$$

$$= \left[\pi^{11-1}(1-\pi)^{7-1}\right]$$

This function is proportional to a beta(11,7) distribution.

# An example with a continuous variable:
## Obtaining the posterior by Bayes Theorem

posterior

$$p(y \mid x^*) = \frac{p(x^* \mid y)p(y)}{\int_y p(x^* \mid y)p(y)\partial y}$$

After normalizing:

$$p(\pi \mid r = 7) = \frac{\pi^{11-1}(1-\pi)^{7-1}}{\int_z \left[z^{11-1}(1-z)^{7-1}\right]\partial z}$$

Now, how can we get an idea of what this means we believe about $\pi$ after combining our prior belief and our observations?

# An example with a continuous variable: In pictures

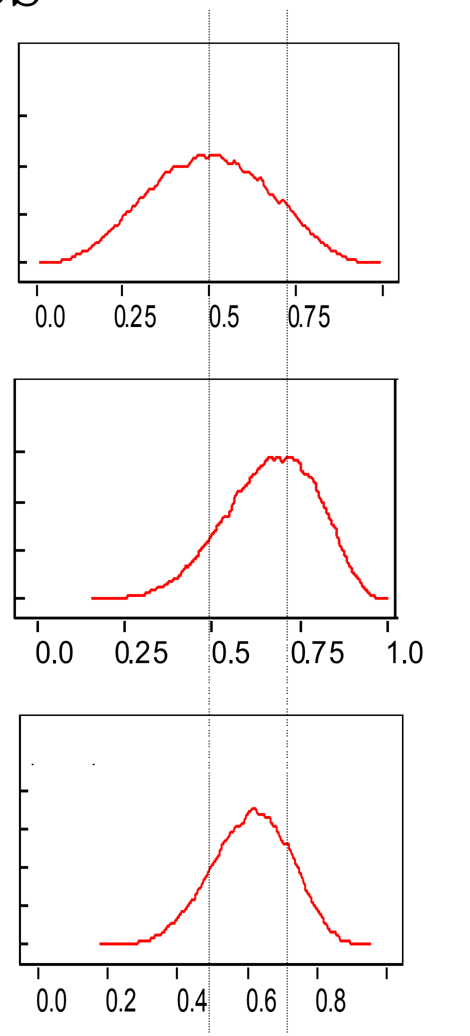
Prior

x

Likelihood

$\propto$

Posterior



# Dirichlet—Categorical conjugate distribution

- Assume a variable $X$ takes on category $1,\ldots,K$ with probabilities $\pi_1,\ldots\pi_K$
- Take $N$ draws from this distribution and observe counts $N=X_1+ \ldots + X_K$
- Likelihood is $p(X_1,,\ldots,X_K) \propto \pi_1^{X_1} \cdots \pi_K^{X_K}$
- Dirichlet Prior: $f(\pi_1,,\ldots,\pi_K) \propto \pi_1^{\alpha_1-1} \cdots \pi_K^{\alpha_K-1}$
- Posterior:

$$f(\pi_1,,\ldots,\pi_K|X_1,\ldots,X_K) \propto \pi_1^{X_1+\alpha_1-1} \cdots \pi_K^{X_K+\alpha_K-1}$$

# Updating an unconditional probability table (no parent variables)

- Prior is a table of alphas:

| | | |
|---|---|---|
| $\alpha_1$ | … | $\alpha_K$ |

- Sum of alphas is pseudo-sample size for prior: Netica calls this Node Experience $A = \alpha_1 + \cdots + \alpha_K$
- Sufficient statistic is a table of counts in each category

| | | |
|---|---|---|
| $X_1$ | … | $X_K$ |

- Posterior is an updated table

| | | |
|---|---|---|
| $\alpha_1+X_1$ | … | $\alpha_K+X_K$ |

- With updated Node Experience $A'=A+N$

# Details

- Equivalent to beta-binomial when variable only takes two values
- Alphas must be positive, but don't need to be integers
- Alpha = ½ is non-informative prior
- A (sum of alphas) acts like a pseudo-sample size for the prior
- Can also write as $\alpha_k = A\pi^*_k$

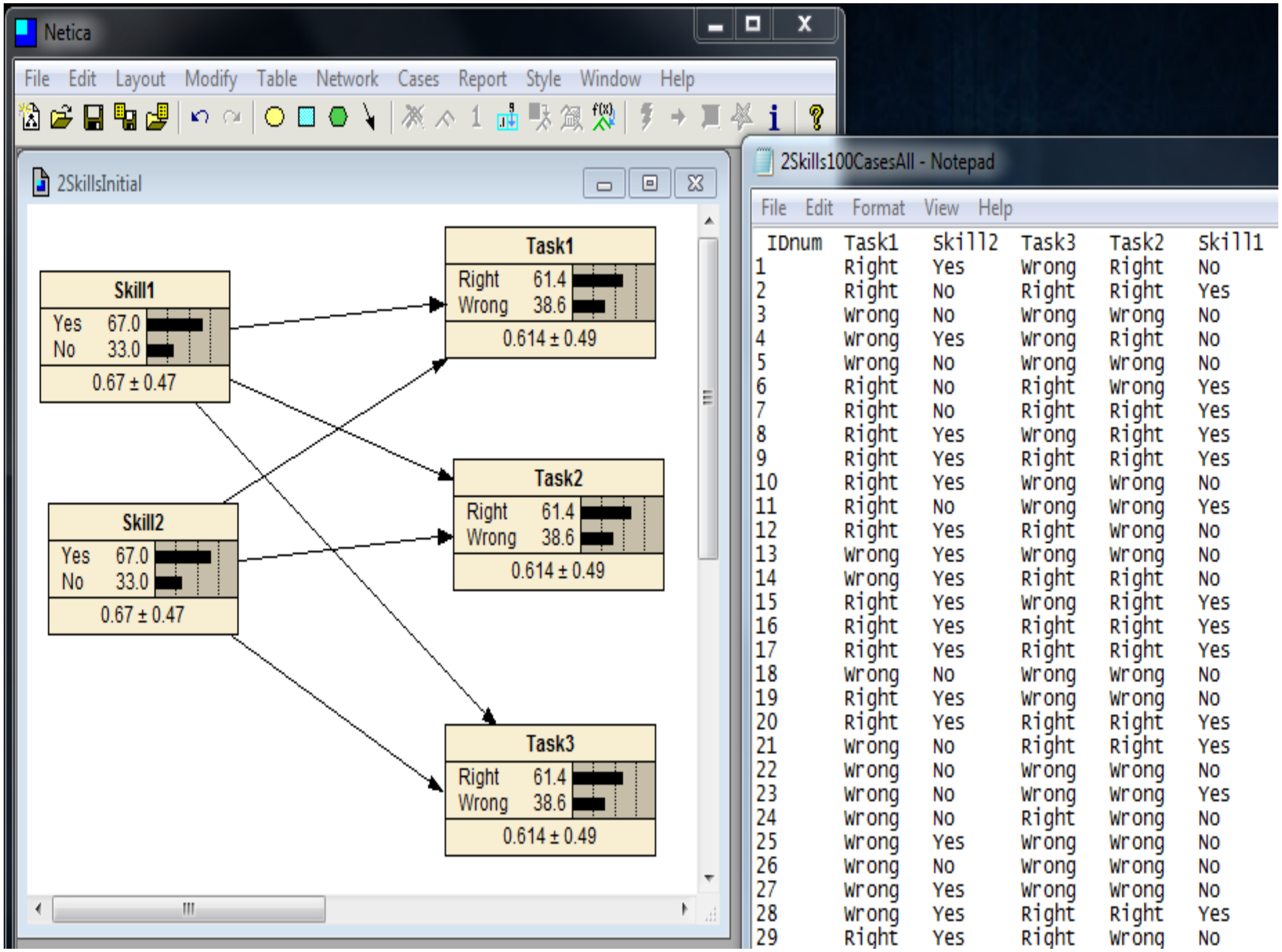
# CPT updating when parents are fully observed

- Data are contingency table of child variable given parents
- Prior is a table of pseudo-counts
- Get posterior by adding them together

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1K} \\ \vdots & \ddots & \vdots \\ \alpha_{J1} & \cdots & \alpha_{JK} \end{pmatrix} + \begin{pmatrix} X_{11} & \cdots & X_{1K} \\ \vdots & \ddots & \vdots \\ X_{J1} & \cdots & X_{JK} \end{pmatrix} = \begin{pmatrix} \alpha_{11}+X_{11} & \cdots & \alpha_{1K}+X_{1K} \\ \vdots & \ddots & \vdots \\ \alpha_{J1}+X_{J1} & \cdots & \alpha_{JK}+X_{JK} \end{pmatrix}$$

Note: Both prior and posterior effective sample size (Node Experience) can be different for each row.

# Netica example – fully observed



# RNetica example (Ex 8.3)

- File Hyperdirchlet
- Set up network
- Two parents, one child

```
sess <- NeticaSession()
startSession(sess)
hdnet <- CreateNetwork("hyperDirchlet”,sess)
skills <- NewDiscreteNode(hdnet,c("Skill1","Skill2"),
      c("High","Medium","Low"))
obs <-
NewDiscreteNode(hdnet,"Observable",c("Right","Wrong"))
NodeParents(obs) <- skills
```

# Set up prior for Observation

- Do this by setting CPT and NodeExperience (row pseudo-sample sizes)

```
ptab <- data.frame(
   Skill1=rep(c("High","Medium","Low"),3),
   Skill2=rep(c("High","Medium","Low"), each=3),
  Right=c(.975,.875,.5,.875,.5,.125,.5,.125,.025),
Wrong=1-c(.975,.875,.5,.875,.5,.125,.5,.125,.025))

obs[] <- ptab
NodeExperience(obs) <- 10 #All rows equally
weighted
```

# Aside: Using CPTtools

- The function calcDPCframe will (among other things) calculate tables according to the DiBello—Samejima models described in the morning session.

```
## Using CPTtools
ptab1 <- calcDPCFrame(
   ParentStates(obs),
   NodeStates(obs),
   log(c(Skill1=1.2,Skill2=.8)),0,
   rules="Compensatory")
```

- Note uses log of discrimination as parameter

# Prior CPT

## ptab

| | Skill1 | Skill2 | Right | Wrong |
|---|---|---|---|---|
| 1 | High | High | 0.975 | 0.025 |
| 2 | Medium | High | 0.875 | 0.125 |
| 3 | Low | High | 0.500 | 0.500 |
| 4 | High | Medium | 0.875 | 0.125 |
| 5 | Medium | Medium | 0.500 | 0.500 |
| 6 | Low | Medium | 0.125 | 0.875 |
| 7 | High | Low | 0.500 | 0.500 |
| 8 | Medium | Low | 0.125 | 0.875 |
| 9 | Low | Low | 0.025 | 0.975 |

## rescaleTable(ptab,10)

| | Skill1 | Skill2 | Right | Wrong |
|---|---|---|---|---|
| 1 | High | High | 9.75 | 0.25 |
| 2 | Medium | High | 8.75 | 1.25 |
| 3 | Low | High | 5.00 | 5.00 |
| 4 | High | Medium | 8.75 | 1.25 |
| 5 | Medium | Medium | 5.00 | 5.00 |
| 6 | Low | Medium | 1.25 | 8.75 |
| 7 | High | Low | 5.00 | 5.00 |
| 8 | Medium | Low | 1.25 | 8.75 |
| 9 | Low | Low | 0.25 | 9.75 |

# Netica Case files

- Text file, column separated by tabs (same as .xls files, but have .cas extension)
- One column for each observed variable (need both parents and child in this case)
- Optional IDnum column
- Optional NumCases column gives replication count
- So can either repeat out cases, or use summary counts.
- write.CaseFile() writes out a case file for use with Netica

# Case Table for Ex 8.3

```
dtab <- data.frame(Skill1=rep(c("High","Medium","Low"),3,each=2),
                   Skill2=rep(c("High","Medium","Low"),each=6),
                   Observable=rep(c("Right","Wrong"),9),
                   NumCases=c(293,3,
                     112,16,
                     0,1,
                     14,1,
                     92,55,
                     4,5,
                     5,1,
                     62,156,
                     8,172))

write.CaseFile(dtab,"Ex8.3.cas")
```

# Example Case File

| | Skill1 | Skill2 | Observable | NumCases |
|---|---|---|---|---|
| 1 | High | High | Right | 293 |
| 2 | High | High | Wrong | 3 |
| 3 | Medium | High | Right | 112 |
| 4 | Medium | High | Wrong | 16 |
| 5 | Low | High | Right | 0 |
| 6 | Low | High | Wrong | 1 |
| 7 | High | Medium | Right | 14 |
| 8 | High | Medium | Wrong | 1 |
| 9 | Medium | Medium | Right | 92 |
| 10 | Medium | Medium | Wrong | 55 |
| 11 | Low | Medium | Right | 4 |
| 12 | Low | Medium | Wrong | 5 |
| 13 | High | Low | Right | 5 |
| 14 | High | Low | Wrong | 1 |
| 15 | Medium | Low | Right | 62 |
| 16 | Medium | Low | Wrong | 156 |
| 17 | Low | Low | Right | 8 |
| 18 | Low | Low | Wrong | 172 |

# Learn CPTs

- LearnCases does complete data hyper-Dirichlet updating

```
LearnCases("Ex8.3.cas",obs)
NodeExperience(obs)
                  Skill2
Skill1     High Medium Low
  High      306     25  16
  Medium    138    157 228
  Low        11     19 190
```

# Prior and Posterior CPTs

## Prior

| | Skill1 | Skill2 | Right | Wrong |
|---|---|---|---|---|
| 1 | High | High | 0.975 | 0.025 |
| 2 | Medium | High | 0.875 | 0.125 |
| 3 | Low | High | 0.500 | 0.500 |
| 4 | High | Medium | 0.875 | 0.125 |
| 5 | Medium | Medium | 0.500 | 0.500 |
| 6 | Low | Medium | 0.125 | 0.875 |
| 7 | High | Low | 0.500 | 0.500 |
| 8 | Medium | Low | 0.125 | 0.875 |
| 9 | Low | Low | 0.025 | 0.975 |

## Posterior

| | Skill1 | Skill2 | Right | Wrong |
|---|---|---|---|---|
| 1 | High | High | 0.989 | 0.011 |
| 2 | Medium | High | 0.848 | 0.152 |
| 3 | Low | High | 0.795 | 0.205 |
| 4 | High | Medium | 0.760 | 0.240 |
| 5 | Medium | Medium | 0.588 | 0.412 |
| 6 | Low | Medium | 0.276 | 0.724 |
| 7 | High | Low | 0.859 | 0.141 |
| 8 | Medium | Low | 0.277 | 0.723 |
| 9 | Low | Low | 0.068 | 0.932 |

# Prior and Posterior Alphas

## Prior

| | Skill1 | Skill2 | Right | Wrong |
|---|---|---|---|---|
| 1 | High | High | 9.75 | 0.25 |
| 2 | Medium | High | 8.75 | 1.25 |
| 3 | Low | High | 5.00 | 5.00 |
| 4 | High | Medium | 8.75 | 1.25 |
| 5 | Medium | Medium | 5.00 | 5.00 |
| 6 | Low | Medium | 1.25 | 8.75 |
| 7 | High | Low | 5.00 | 5.00 |
| 8 | Medium | Low | 1.25 | 8.75 |
| 9 | Low | Low | 0.25 | 9.75 |

## Posterior

| | Skill1 | Skill2 | Right | Wrong |
|---|---|---|---|---|
| 1 | High | High | 302.75 | 3.25 |
| 2 | Medium | High | 117.00 | 21.00 |
| 3 | Low | High | 8.75 | 2.25 |
| 4 | High | Medium | 19.00 | 6.00 |
| 5 | Medium | Medium | 92.25 | 64.75 |
| 6 | Low | Medium | 5.25 | 13.75 |
| 7 | High | Low | 13.75 | 2.25 |
| 8 | Medium | Low | 63.25 | 164.75 |
| 9 | Low | Low | 13.00 | 177.00 |

# Problems with hyper-Dirichlet approach

- Learn more about some rows than others
- Local parameter independence assumption is unrealistic – often want CPT to be monotonic (increasing skill means increasing chance of success)
  - $\lambda_{2,2} > \lambda_{2,1} > \lambda_{1,1}$ and $\lambda_{2,2} > \lambda_{1,2} > \lambda_{1,1}$
- Solution is to use parametric models for CPT:
  - Noisy-min & Noisy-max
  - DiBello-Samejima families
  - Discrete Partial Credit families

# Learning CPTs for a parametric family

- Contingency table is sufficient statistic for law for any CPT!
- Pick value of law parameters that maximize the posterior probability (or likelihood) of the observed contingency table.
- Fully Bayesian method
  - Put hyper-laws over law hyperparameters
  - Calculate observed contingency table
  - MAP estimates maximize posterior probability of contingency table
- Semi-Bayesian method
  - Use prior hyperparameters to calculate prior table.
  - Establish a pseudo-sample size for each row and calculate prior alphas
  - Do hyper-Dirichlet updating to get posterior alphas
  - MAP estimates maximize posterior probability of posterior alphas (treating them as if they were data)
  - CPTtools function `mapCPT` does this

# Latent and Missing Values

- These are okay as long as they are *missing at random*
- MAR means missingness indicator is conditionally independent of the value of the missing variable given the fully observed variables
- Latent variables are always MCAR
- With other missing variables, it depends on the study design
- Can use the EM or MCMC algorithms in the presence of MAR data

# EM Algorithm (Dempster, Laird & Rubin, 1977)

Key idea:

1. Pick a set of value for parameters
2. *E-step (a):* Calculate distribution for missing variables given observed variables & current parameter values.
3. *E-step (b):* Calculate expected value of sufficient statistics
4. *M-step:* Use Gradient Decent to produce MAP/MLE estimates for parameters given sufficient statistics
5. Loop 2—4 until convergence

# EM algorithm details

- Only need to take a few steps of the gradient algorithm in Step 4 (Generalized EM)
- Can exploit conditional independence conditions, particularly global parameter independence (Structual EM, Meng and van Dyke)
  - Once CPT at a time
- Can be slow
  - But not as slow as MCMC
- Netica provides built-in support for special case of hyper-Dirichlet law

# Expected value of missing (latent) node

- Can calculate this using ordinary Netica operations (instantiate all observed variables and read off joint beliefs)
- Instead of adding count to the table, add fractional count to the table
- Similarly use joint beliefs when more than one parent is missing

# Example

- Observable $X$ in $\{0, 1\}$; Latent $\theta$ in $\{H,M,L\}$
- Observations:
  1. $X=1$; $p(\theta) = H{:}.33,\ M{:}.33,\ L{:}.33$
  2. $X=1$; $p(\theta) = H{:}.5,\ M{:}.33,\ L{:}.2$
  3. $X=0$; $p(\theta) = H{:}.2,\ M{:}.3,\ L{:}.5$
- Expected table:

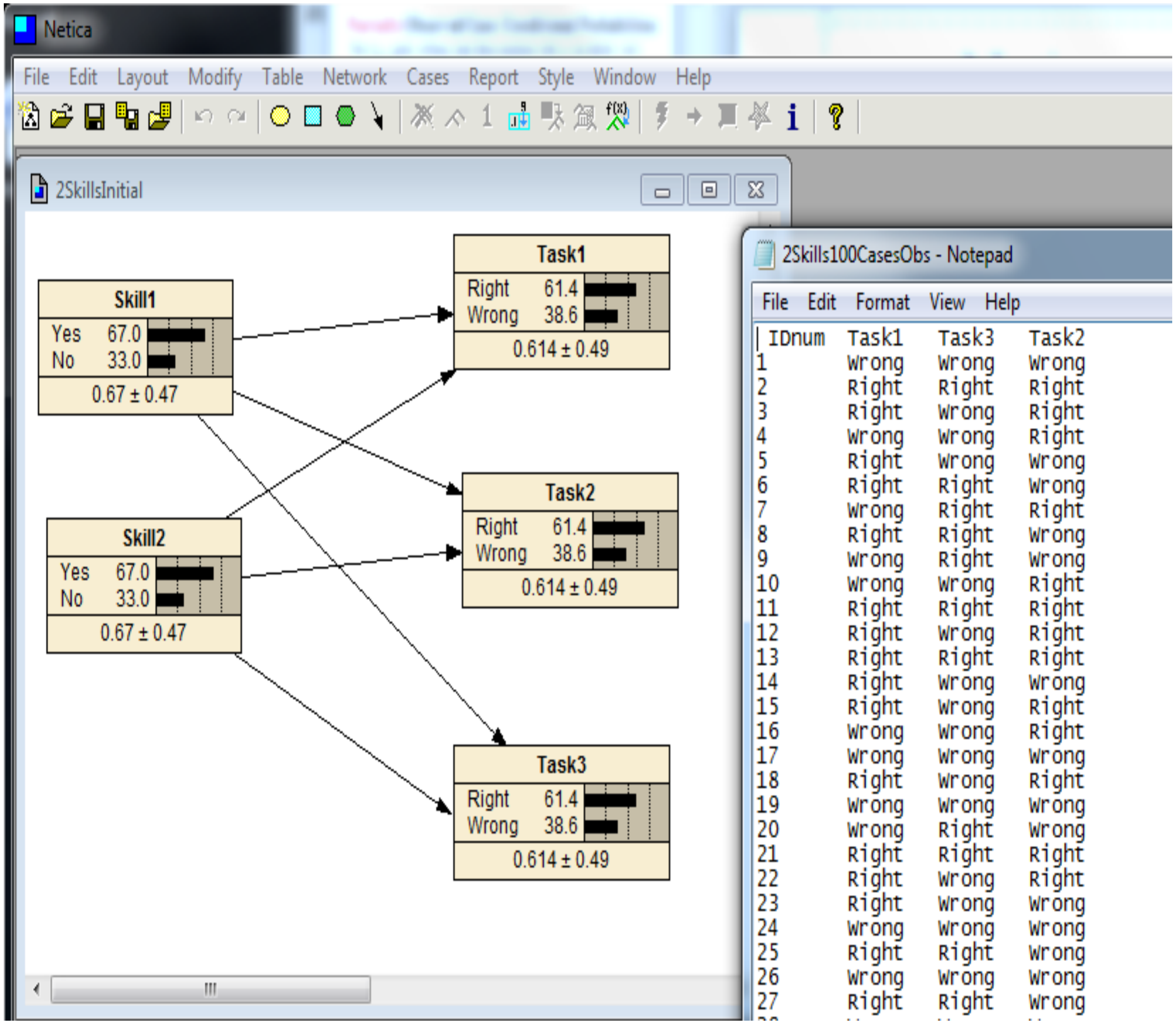
| | H | M | L |
|---|---|---|---|
| 1 | .83 | .67 | .53 |
| 0 | .2 | .3 | .5 |

# EM for hyper-Dirichlet (RNetica LearnCPTs function)

1. Use current CPTs to calculate expected tables for all of the CPTs we are learning
2. Use the hyper-Dirichlet conjugate updating to update the CPTs
3. Loop 1 and 2 until convergence

*Note: RNetica LearnCPT function currently does not reveal whether or not convergence was reached.*

# Netica example – partially observed



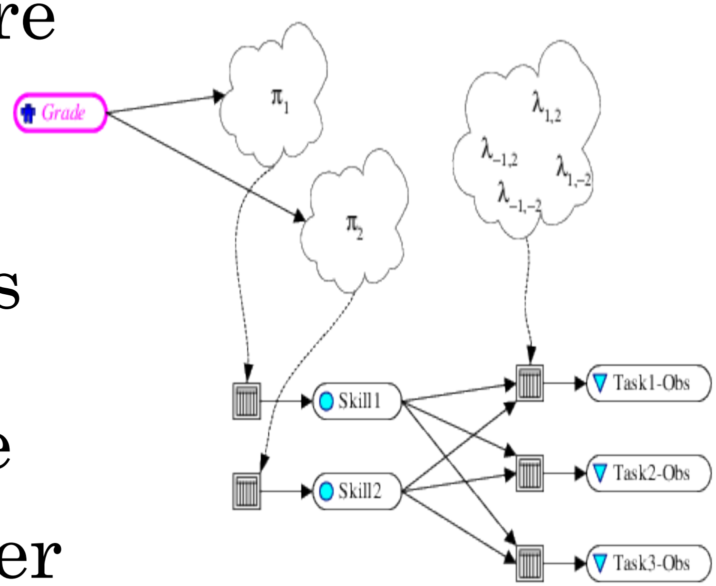
# Parameterized tables

1. Use current parameters to set initial CPTs
2. Use Netica's LearnCPTs to calculate posterior tables
3. Multiple posterior tables by node experience to get pseudo-table for each CPT
4. Use gradient decent to optimize CPT parameters
5. Loop 1—4 until convergence

I'm currently working on an implementation in R (Peanut package function `GEMfit`; available from RNetica site).

# Breakdown of global parameter independence

- Even if parameters are *a priori* independent, when there is missing (or latent) data then parameters are not independent *a posteriori*.
- EM algorithm only gives point estimate, does not capture this dependence
- There might also be other information which makes parameters dependent.



# Markov Chain Monte Carlo (MCMC)

- In place of E-step, randomly sample values for unknown (latent & missing) variables
- In place of M-step, randomly sample values for parameters
- Takes longer than EM, but gives you an impression of the whole distribution rather than just a part.