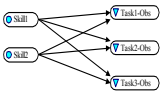
Learning CPTs

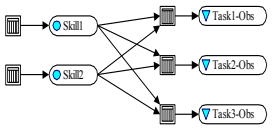
Thanks to Bob Mislevy for letting me use some of the slides from his class.

First Layer



- A simple model with two skills and 3 observables

Distributions and Variables



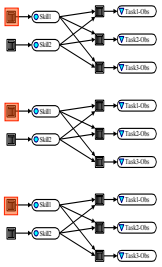
- Variables (values are person specific)
- *Distributions* provide probabilities for variables

Different People, Same Distributions

Student 1

Student 2

Student 3



Different People, Same Distributions

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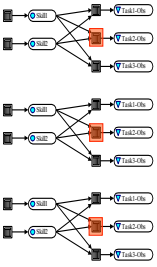
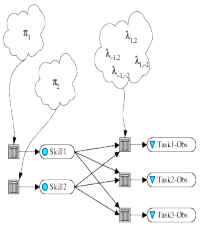
Student 3



Second Layer

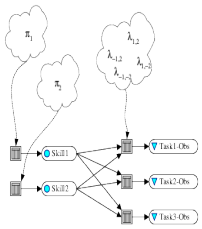
- Distributions have Parameters
- Parameters are the same across all people
- Parameters drop down into first layer to do person specific computations (e.g., scoring)



Probability distributions of parameters are called *Laws*

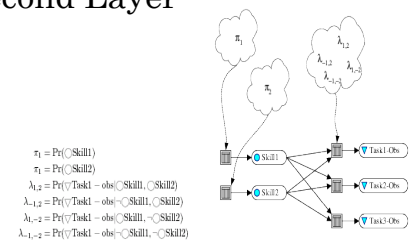
Second Layer

- Distributions have Parameters
- Parameters are the same across all people
- Parameters drop down into first layer to do person specific computations (e.g., scoring)



Probability distributions of parameters are called *Laws*

Second Layer



Speigelhalter And Lauritzen (1990)

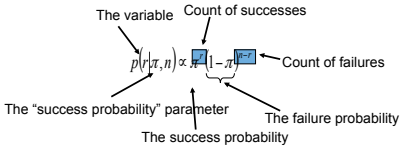
- *Global Parameter Independence* – parameters of laws for different CPTs are independent
- *Local Parameter Independence* – parameters for laws for different rows of CPTs are independent

Under these two assumptions, the natural conjugate law of a Bayesian network is a *hyper-Dirichlet law*, a law where the probabilities on each row of each CPT follow a Dirichlet law.

Abusing the definition, we say that a CPT for which each rows is given an independent Dirichlet law follows a *hyper-Dirichlet distribution (*really should be law).

A closer look at the binomial distribution

- **Binomial.** For counts of successes in binary trials, each with probability p, in n independent trials. E.g., n coin flips, with p the common probability of heads.



We will be using this as a likelihood in an example of the use of conjugate distribution

A closer look at the Beta distribution

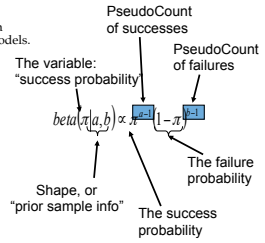
- **Beta.** Defined on [0,1]. Conjugate prior for the probability parameter in Bernoulli & binomial models.

```
p ~ dbeta(a,b)
```

Mean(p) $\frac{a}{a+b}$

Variance(p) $\frac{ab}{(a+b)^2(a+b+1)}$

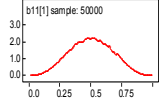
Mode(p) $\frac{a-1}{a+b-2}$



An example with a continuous variable:
A beta-binomial example--the Prior Distribution

- The prior distribution:

  Let's suppose we think it is more likely that the coin is close to fair, so π is probably nearer to .5 than it is to either 0 or 1. We don't have any reason to think it is biased toward either heads or tails, so we'll want a prior distribution that is symmetric around .5. We're not real sure about what π might be--say about as sure as only 6 observations. This corresponds to 3 pseudo-counts of H and 3 of T, which, if we want to use a beta distribution to express this belief, corres



An example with a continuous variable:
A beta-binomial example--the Prior Distribution

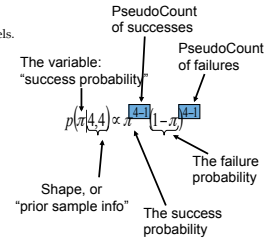
- **Beta.** Defined on [0,1]. Conjugate prior for the probability parameter in Bernoulli & binomial models.

```
π ~ dbeta(4,4)
```

Mean(π): $\frac{4}{4+4} = .5$

Variance(π): $\frac{4 \cdot 4}{(4+4)^2(4+4+1)} = .028$

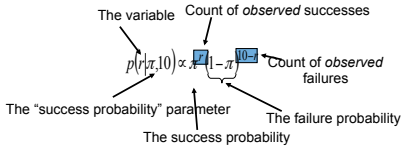
Mode(π): $\frac{4-1}{4+4-2} = .5$



An example with a continuous variable:
A beta-binomial example--the Likelihood

- The likelihood:

  Next we will flip the coin ten times. Assuming the same true (but unknown to us) value of π is in effect for each of ten independent trials, we can use the binomial distribution to model the probability of getting any number of heads: i.e.,

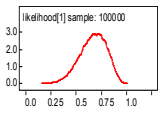


An example with a continuous variable:
A beta-binomial example--the Likelihood

- The likelihood:

  We flip the coin ten times, and observe 7 heads; i.e., r=7. The likelihood is obtained now using the same form as in the preceding slide, except now r is fixed at 7 and we are interested in the relative value of this function at different possible values of π:

$$p(7|\pi,10) \propto \pi^{7}(1-\pi)^{3}$$



An example with a continuous variable:
Obtaining the posterior by Bayes Theorem

General form:

posterior likelihood prior

$$p(y|x^*) \propto p(x^*|y)\ p(y)$$

In our example, 7 plays the role of x^*, and p plays the role of y. Before normalizing:

$$p(\pi|r=7) \propto \left[\pi^{7}(1-\pi)^{3}\right]\left[\pi^{4-1}(1-\pi)^{4-1}\right]$$

$$= \left[\pi^{10}(1-\pi)^{6}\right]$$

$$= \left[\pi^{11-1}(1-\pi)^{7-1}\right]$$

This function is proportional to a beta(11,7) distribution.

An example with a continuous variable:
Obtaining the posterior by Bayes Theorem

posterior

$$p(y|x^*) = \frac{p(x^*|y)p(y)}{\int_y p(x^*|y)p(y)\partial y}$$

After normalizing:

$$p(\pi|r=7) = \frac{\pi^{11-1}(1-\pi)^{7-1}}{\int_z \left[z^{11-1}(1-z)^{7-1}\right]\partial z}$$

Now, how can we get an idea of what this means we believe about π after combining our prior belief and our observations?

An example with a continuous variable: In pictures

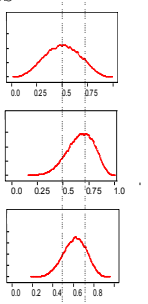
Prior

x

Likelihood

=

Posterior



Dirichlet—Categorical conjugate distribution

- Assume a variable X takes on category $1,\ldots,K$ with probabilities $\pi_1,\ldots\pi_K$
- Take N draws from this distribution and observe counts $N=X_1+\ldots+X_K$
- Likelihood is $p(X_1,,\ldots,X_K) \propto \pi_1^{X_1}\cdots\pi_K^{X_K}$
- Dirichlet Prior: $f(\pi_1,,\ldots,\pi_K) \propto \pi_1^{\alpha_1-1}\cdots\pi_K^{\alpha_K-1}$
- Posterior:

$$f(\pi_1,,\ldots,\pi_K|X_1,\ldots,X_K) \propto \pi_1^{X_1+\alpha_1-1}\cdots\pi_K^{X_K+\alpha_K-1}$$

Updating an unconditional probability table (no parent variables)

- Prior is a table of alphas:

α_1	...	α_K

- Sum of alphas is pseudo-sample size for prior: Netica calls this Node Experience $A=\alpha_1+\cdots+\alpha_K$
- Sufficient statistic is a table of counts in each category

X_1	...	X_K

- Posterior is an updated table

α_1+X_1	...	α_K+X_K

- With updated Node Experience $A'=A+N$

Details

- Equivalent to beta-binomial when variable only takes two values
- Alphas must be positive, but don't need to be integers
- Alpha = ½ is non-informative prior
- A (sum of alphas) acts like a pseudo-sample size for the prior $\alpha_k = A\pi_k^*$
- Can also write as

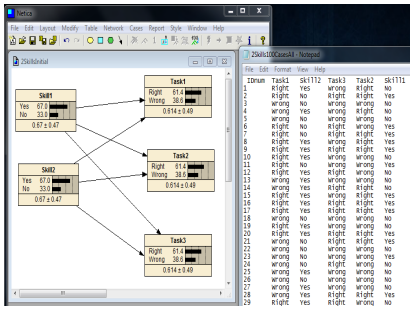
CPT updating when parents are fully observed

- Data are contingency table of child variable given parents
- Prior is a table of pseudo-counts
- Get posterior by adding them together

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1K} \\ \vdots & \ddots & \vdots \\ \alpha_{J1} & \cdots & \alpha_{JK} \end{pmatrix} + \begin{pmatrix} X_{11} & \cdots & X_{1K} \\ \vdots & \ddots & \vdots \\ X_{J1} & \cdots & X_{JK} \end{pmatrix} = \begin{pmatrix} \alpha_{11}+X_{11} & \cdots & \alpha_{1K}+X_{1K} \\ \vdots & \ddots & \vdots \\ \alpha_{J1}+X_{J1} & \cdots & \alpha_{JK}+X_{JK} \end{pmatrix}$$

Note: Both prior and posterior effective sample size (Node Experience) can be different for each row.

Netica example – fully observed



RNetica example (Ex 8.3)

- File Hyperdirchlet
- Set up network
- Two parents, one child

```
sess <- NeticaSession()
startSession(sess)
hdnet <- CreateNetwork("hyperDirchlet",sess)
skills <- NewDiscreteNode(hdnet,c("Skill1","Skill2"),
      c("High","Medium","Low"))
obs <-
NewDiscreteNode(hdnet,"Observable",c("Right","Wrong"))
NodeParents(obs) <- skills
```

Set up prior for Observation

- Do this by setting CPT and NodeExperience (row pseudo-sample sizes)

```
ptab <- data.frame(
    Skill1=rep(c("High","Medium","Low"),3),
    Skill2=rep(c("High","Medium","Low"), each=3),
  Right=c(.975,.875,.5,.875,.5,.125,.5,.125,.025),
Wrong=1-c(.975,.875,.5,.875,.5,.125,.5,.125,.025))

obs[] <- ptab
NodeExperience(obs) <- 10 #All rows equally
weighted
```


Aside: Using CPTtools

- The function calcDPCframe will (among other things) calculate tables according to the DiBello—Samejima models described in the morning session.

```
## Using CPTtools
ptab1 <- calcDPCFrame(
    ParentStates(obs),
    NodeStates(obs),
    log(c(Skill1=1.2,Skill2=.8)),0,
    rules="Compensatory")
```

- Note uses log of discrimination as parameter

Prior CPT

ptab

	Skill1	Skill2	Right	Wrong
1	High	High	0.975	0.025
2	Medium	High	0.875	0.125
3	Low	High	0.500	0.500
4	High	Medium	0.875	0.125
5	Medium	Medium	0.500	0.500
6	Low	Medium	0.125	0.875
7	High	Low	0.500	0.500
8	Medium	Low	0.125	0.875
9	Low	Low	0.025	0.975

rescaleTable(ptab,10)

	Skill1	Skill2	Right	Wrong
1	High	High	9.75	0.25
2	Medium	High	8.75	1.25
3	Low	High	5.00	5.00
4	High	Medium	8.75	1.25
5	Medium	Medium	5.00	5.00
6	Low	Medium	1.25	8.75
7	High	Low	5.00	5.00
8	Medium	Low	1.25	8.75
9	Low	Low	0.25	9.75

Netica Case files

- Text file, column separated by tabs (same as .xls files, but have .cas extension)
- One column for each observed variable (need both parents and child in this case)
- Optional IDnum column
- Optional NumCases column gives replication count
- So can either repeat out cases, or use summary counts.
- write.CaseFile() writes out a case file for use with Netica

Case Table for Ex 8.3

```
dtab <- data.frame(Skill1=rep(c("High","Medium","Low"),3,each=2),
                   Skill2=rep(c("High","Medium","Low"),each=6),
                   Observable=rep(c("Right","Wrong"),9),
                   NumCases=c(293,3,
                     112,16,
                     0,1,
                     14,1,
                     92,55,
                     4,5,
                     5,1,
                     62,156,
                     8,172))

write.CaseFile(dtab,"Ex8.3.cas")
```


Example Case File

	Skill1	Skill2	Observable	NumCases
1	High	High	Right	293
2	High	High	Wrong	3
3	Medium	High	Right	112
4	Medium	High	Wrong	16
5	Low	High	Right	0
6	Low	High	Wrong	1
7	High	Medium	Right	14
8	High	Medium	Wrong	1
9	Medium	Medium	Right	92
10	Medium	Medium	Wrong	55
11	Low	Medium	Right	4
12	Low	Medium	Wrong	5
13	High	Low	Right	5
14	High	Low	Wrong	1
15	Medium	Low	Right	62
16	Medium	Low	Wrong	156
17	Low	Low	Right	8
18	Low	Low	Wrong	172

Learn CPTs

- LearnCases does complete data hyper-Dirichlet updating

```
LearnCases("Ex8.3.cas",obs)
NodeExperience(obs)
                Skill2
Skill1    High Medium Low
  High     306     25  16
  Medium   138    157 228
  Low       11     19 190
```


Prior and Posterior CPTs

Prior

	Skill1	Skill2	Right	Wrong
1	High	High	0.975	0.025
2	Medium	High	0.875	0.125
3	Low	High	0.500	0.500
4	High	Medium	0.875	0.125
5	Medium	Medium	0.500	0.500
6	Low	Medium	0.125	0.875
7	High	Low	0.500	0.500
8	Medium	Low	0.125	0.875
9	Low	Low	0.025	0.975

Posterior

	Skill1	Skill2	Right	Wrong
1	High	High	0.989	0.011
2	Medium	High	0.848	0.152
3	Low	High	0.795	0.205
4	High	Medium	0.760	0.240
5	Medium	Medium	0.588	0.412
6	Low	Medium	0.276	0.724
7	High	Low	0.859	0.141
8	Medium	Low	0.277	0.723
9	Low	Low	0.068	0.932

Prior and Posterior Alphas

Prior

	Skill1	Skill2	Right	Wrong
1	High	High	9.75	0.25
2	Medium	High	8.75	1.25
3	Low	High	5.00	5.00
4	High	Medium	8.75	1.25
5	Medium	Medium	5.00	5.00
6	Low	Medium	1.25	8.75
7	High	Low	5.00	5.00
8	Medium	Low	1.25	8.75
9	Low	Low	0.25	9.75

Posterior

	Skill1	Skill2	Right	Wrong
1	High	High	302.75	3.25
2	Medium	High	117.00	21.00
3	Low	High	8.75	2.25
4	High	Medium	19.00	6.00
5	Medium	Medium	92.25	64.75
6	Low	Medium	5.25	13.75
7	High	Low	13.75	2.25
8	Medium	Low	63.25	164.75
9	Low	Low	13.00	177.00

Problems with hyper-Dirichlet approach

- Learn more about some rows than others
- Local parameter independence assumption is unrealistic – often want CPT to be monotonic (increasing skill means increasing chance of success)
  - $\lambda_{2,2} > \lambda_{2,1} > \lambda_{1,1}$ and $\lambda_{2,2} > \lambda_{1,2} > \lambda_{1,1}$
- Solution is to use parametric models for CPT:
  - Noisy-min & Noisy-max
  - DiBello-Samejima families
  - Discrete Partial Credit families

Learning CPTs for a parametric family

- Contingency table is sufficient statistic for law for any CPT!
- Pick value of law parameters that maximize the posterior probability (or likelihood) of the observed contingency table.
- Fully Bayesian method
  - Put hyper-laws over law hyperparameters
  - Calculate observed contingency table
  - MAP estimates maximize posterior probability of contingency table
- Semi-Bayesian method
  - Use prior hyperparameters to calculate prior table.
  - Establish a pseudo-sample size for each row and calculate prior alphas
  - Do hyper-Dirichlet updating to get posterior alphas
  - MAP estimates maximize posterior probability of posterior alphas (treating them as if they were data)
  - CPTtools function `mapCPT` does this

Latent and Missing Values

- These are okay as long as they are *missing at random*
- MAR means missingness indicator is conditionally independent of the value of the missing variable given the fully observed variables
- Latent variables are always MCAR
- With other missing variables, it depends on the study design
- Can use the EM or MCMC algorithms in the presence of MAR data

EM Algorithm (Dempster, Laird & Rubin, 1977)

Key idea:

1. Pick a set of value for parameters
2. *E-step (a):* Calculate distribution for missing variables given observed variables & current parameter values.
3. *E-step (b):* Calculate expected value of sufficient statistics
4. *M-step:* Use Gradient Decent to produce MAP/ MLE estimates for parameters given sufficient statistics
5. Loop 2—4 until convergence

EM algorithm details

- Only need to take a few steps of the gradient algorithm in Step 4 (Generalized EM)
- Can exploit conditional independence conditions, particularly global parameter independence (Structual EM, Meng and van Dyke)
  - Once CPT at a time
- Can be slow
  - But not as slow as MCMC
- Netica provides built-in support for special case of hyper-Dirichlet law

Expected value of missing (latent) node

- Can calculate this using ordinary Netica operations (instantiate all observed variables and read off joint beliefs)
- Instead of adding count to the table, add fractional count to the table
- Similarly use joint beliefs when more than one parent is missing

Example

- Observable X in *{0, 1};* Latent θ in *{H,M,L}*
- Observations:
  1. *X=1; p(θ) = H:.33, M:.33, L:.33*
  2. *X=1; p(θ) = H:.5, M:.33, L:.2*
  3. *X=0; p(θ) = H:.2, M:.3, L:.5*
- Expected table:

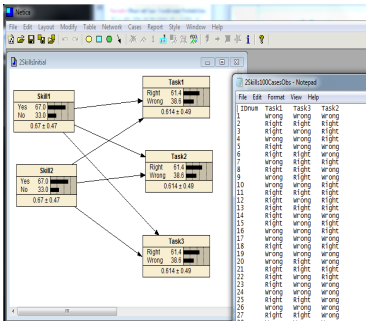
	H	M	L
1	.83	.67	.53
0	.2	.3	.5

EM for hyper-Dirichlet (RNetica LearnCPTs function)

1. Use current CPTs to calculate expected tables for all of the CPTs we are learning
2. Use the hyper-Dirichlet conjugate updating to update the CPTs
3. Loop 1 and 2 until convergence

Note: RNetica LearnCPT function currently does not reveal whether or not convergence was reached.

Netica example – partially observed



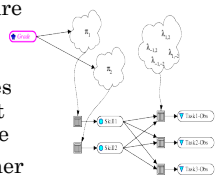
Parameterized tables

1. Use current parameters to set initial CPTs
2. Use Netica's LearnCPTs to calculate posterior tables
3. Multiple posterior tables by node experience to get pseudo-table for each CPT
4. Use gradient decent to optimize CPT parameters
5. Loop 1—4 until convergence

I'm currently working on an implementation in R (Peanut package function `GEMfit`; available from RNetica site).

Breakdown of global parameter independence

- Even if parameters are *a priori* independent, when there is missing (or latent) data then parameters are not independent *a posteriori*.
- EM algorithm only gives point estimate, does not capture this dependence
- There might also be other information which makes parameters dependent.



Markov Chain Monte Carlo (MCMC)

- In place of E-step, randomly sample values for unknown (latent & missing) variables
- In place of M-step, randomly sample values for parameters
- Takes longer than EM, but gives you an impression of the whole distribution rather than just a part.